

HOMOLOGOUS NON-ISOTOPIC SYMPLECTIC TORI IN HOMOTOPY RATIONAL ELLIPTIC SURFACES

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ABSTRACT. Let $E(1)_K$ denote the homotopy rational elliptic surface corresponding to a knot K in S^3 constructed by R. Fintushel and R.J. Stern in [FS1]. We construct an infinite family of homologous non-isotopic symplectic tori representing a primitive 2-dimensional homology class in $E(1)_K$ when K is any nontrivial fibred knot in S^3 . We also show how these tori can be non-isotopically embedded as homologous symplectic submanifolds in other symplectic 4-manifolds.

1. INTRODUCTION

This paper is a continuation of studies initiated in [EP1] and [EP2] regarding infinite families of non-isotopic and symplectic tori representing the same homology class in a symplectic 4-manifold. Let $E(1)_K$ denote the closed 4-manifold that is homotopy equivalent (hence homeomorphic) to the rational elliptic surface $E(1) \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$ and is obtained by performing knot surgery (as defined in [FS1]) on the rational elliptic surface using a knot K in S^3 . Our main result is the following:

Theorem 1.1. *Let $K \subset S^3$ be a nontrivial fibred knot. Then there exists an infinite family of pairwise non-isotopic symplectic tori representing the primitive homology class $[F] = [T_m]$ in $E(1)_K$, where $[F]$ is the homology class of the fiber in a rational elliptic surface $E(1) \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$.*

Examples of homologous, non-isotopic, symplectic tori were first constructed in [FS2] and then in [EP1], [EP2], [V2] and [V4] (also see [FS3] and [V3]). Recall that infinite families of non-isotopic symplectic tori representing $n[F] \in H_2(E(1)_K)$, $n \geq 2$, were constructed in [EP1]. The family of tori we construct in this paper is in some sense the ‘simplest’ example known so far, when measured in terms of the ‘geography size’ of the ambient (simply-connected) symplectic 4-manifold, the divisibility of the homology class represented, and the complexity of the knotting of the tori. In [V4], using a different technique, Vidussi already constructed symplectic tori representing the same primitive class in $E(1)_K$ for some particular fibred K , namely the trefoil and other fibred knots that have the trefoil as one of their connected summands.

It should be noted that the non-existence of such an infinite family of tori in \mathbb{CP}^2 and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ is proved by Sikorav in [Si] and by Siebert and Tian in [ST], respectively. It is also conjectured that there is at most one symplectic torus (up to isotopy) representing each homology class in $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ for $n < 9$.

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The proof of Theorem 1.1 is spread out over the next three sections. We will review the relevant definitions in Section 2. In Section 5, we will present a direct generalization in the form of Proposition 5.1. In this introduction and elsewhere in the paper by isotopy we mean smooth isotopy and all homology groups have \mathbb{Z} coefficients.

2. LINK SURGERY 4-MANIFOLDS

In this section, first we review the generalization of the link surgery construction of Fintushel and Stern [FS1] by Vidussi [V1], and then give specific link surgeries that will be used in the following sections.

For an n -component link $L \subset S^3$, choose an ordered homology basis of simple closed curves $\{(\alpha_i, \beta_i)\}_{i=1}^n$ such that the pair (α_i, β_i) lie in the i -th boundary component of the link exterior and the intersection of α_i and β_i is 1. Let X_i ($i = 1, \dots, n$) be a 4-manifold containing a 2-dimensional torus submanifold F_i of self-intersection 0. Choose a Cartesian product decomposition $F_i = C_1^i \times C_2^i$, where each $C_j^i \cong S^1$ ($j = 1, 2$) is an embedded circle in X_i .

Definition 2.1. The ordered collection

$$\mathfrak{D} = (\{(\alpha_i, \beta_i)\}_{i=1}^n, \{(X_i, F_i = C_1^i \times C_2^i)\}_{i=1}^n)$$

is called a *link surgery gluing data* for an n -component link L . We define the *link surgery manifold corresponding to \mathfrak{D}* to be the closed 4-manifold

$$L(\mathfrak{D}) := \left[\coprod_{i=1}^n X_i \setminus \nu F_i \right] \bigcup_{F_i \times \partial D^2 = (S^1 \times \alpha_i) \times \beta_i} [S^1 \times (S^3 \setminus \nu L)],$$

where ν denotes the tubular neighbourhoods. Here, the gluing diffeomorphisms between the boundary 3-tori identify the torus $F_i = C_1^i \times C_2^i$ of X_i with $S^1 \times \alpha_i$ factorwise, and act as the complex conjugation on the last remaining S^1 factor.

Remark 2.2. Strictly speaking, the diffeomorphism type of the link surgery manifold $L(\mathfrak{D})$ may possibly depend on the chosen trivialization of $\nu F_i \cong F_i \times D^2$ (the framing of F_i). However, we will suppress this dependence in our notation. It is well known (see e.g. [GS]) that the diffeomorphism type of $L(\mathfrak{D})$ is independent of the framing of F_i when $(X_i, F_i) = (E(1), F)$.

We fix a Cartesian product decomposition $F = C_1 \times C_2$ in $E(1)$. Let K be a knot in S^3 , and let M_K denote the 3-manifold that is the result of the 0-framed surgery on K . Fix a meridian circle $m = \mu(K)$ in M_K .

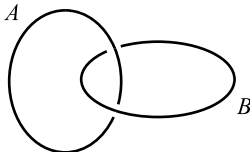


FIGURE 1. Hopf link $L = A \cup B$

Definition 2.3. Let $L \subset S^3$ be the Hopf link in Figure 1. For the link surgery gluing data

$$(2.1) \quad \mathfrak{D} := (\{(\mu(A), \lambda(A)), (\lambda(B), -\mu(B))\}, \{(X_1, F_1 = C_1^1 \times C_2^1), (S^1 \times M_K, T_m = S^1 \times m)\}),$$

we shall denote $L(\mathfrak{D})$ by $(X_1)_K$. Here, $\mu(\cdot)$ and $\lambda(\cdot)$ denote the meridian and the longitude of a knot, respectively. In particular, when $(X_1, F_1 = C_1^1 \times C_2^1) = (E(1), F = C_1 \times C_2)$, we denote $L(\mathfrak{D})$ by $E(1)_K$. This notation is consistent with that of Fintushel and Stern in [FS1] as there is a diffeomorphism between our $L(\mathfrak{D})$ and their fiber sum $E(1)_K = E(1) \#_{F=T_m} (S^1 \times M_K)$.

Note that there is a canonical framing of T_m in $(S^1 \times M_K)$ given by the minimal genus Seifert surface of the knot K . We shall always use this framing to trivialize νT_m .

Lemma 2.4. *If $K \subset S^3$ is a fibred knot, then $E(1)_K$ is a symplectic 4-manifold.*

Proof. This is because there exists a fiber bundle $(S^1 \times M_K) \rightarrow T^2$ when K is fibred, so $(S^1 \times M_K)$ admits a symplectic form with respect to which T_m is a symplectic submanifold (cf. [Th]). Hence we may express $E(1)_K$ as a *symplectic* fiber sum $E(1) \#_{F=T_m} (S^1 \times M_K)$ along symplectic submanifolds F and T_m (cf. [Go]). \square

Lemma 2.5. *The homology class $[F] = [T_m] \in H_2(E(1)_K)$ is primitive.*

Proof. Since $[\mu(A)] = [\lambda(B)] \in H_1(S^3 \setminus \nu L)$, we must have $[S^1 \times \mu(A)] = [S^1 \times \lambda(B)]$ in $H_2(S^1 \times (S^3 \setminus \nu L))$, and so $[F] = [T_m]$ in $H_2(E(1)_K)$. Let Σ denote a Seifert surface of K . Let Σ_K denote a closed surface in $E(1)_K$ that is the internal tubular sum of a punctured section in $[E(1) \setminus \nu F]$ and a punctured surface $\{\text{point}\} \times \Sigma$, glued together along K . Then we have $[\Sigma_K] \cdot [T_m] = \pm 1$. \square

3. FAMILY OF HOMOLOGOUS SYMPLECTIC TORI IN $E(1)_K$

Let $T_C := S^1 \times C \subset [S^1 \times (S^3 \setminus \nu L)] \subset E(1)_K$, where the closed curve $C \subset (S^3 \setminus \nu L)$ is given by Figure 2.

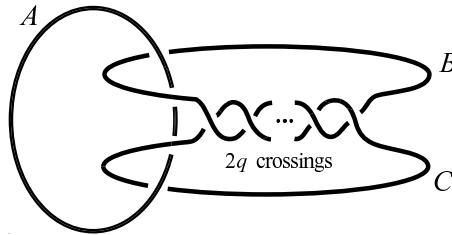


FIGURE 2. 3-component link $L_q = A \cup B \cup C$ in S^3

Lemma 3.1. *If K is a fibred knot, then $T_C = S^1 \times C$ is a symplectic submanifold of $E(1)_K$ and we have $[T_C] = [F]$ in $H_2(E(1)_K)$.*

Proof. It is easy to see that the link exterior $Y := (S^3 \setminus \nu L)$ is diffeomorphic to $S^1 \times \mathbb{A}$, where $\mathbb{A} \cong S^1 \times [0, 1]$ is an annulus. Hence we have

$$[S^1 \times (S^3 \setminus \nu L)] \cong [S^1 \times (S^1 \times \mathbb{A})] \cong T^3 \times [0, 1].$$

We may assume that the symplectic form on $E(1)_K$ restricts to

$$\omega = dx \wedge dy + r dr \wedge d\theta$$

on $[S^1 \times (S^1 \times \mathbb{A})]$, where x and y are the angular coordinates on the first and the second S^1 factors respectively, and (r, θ) are the polar coordinates on the annulus \mathbb{A} . We can embed the curve C inside $(S^1 \times \mathbb{A})$ such that C is transverse to every annulus of the form, $\{\text{point}\} \times \mathbb{A}$, and the restriction $dy|_C$ never vanishes. It follows that $\omega|_{T_C} = (dx \wedge dy)|_{T_C} \neq 0$, and consequently T_C is a symplectic submanifold of $E(1)_K$.

To determine the homology class of T_C , note that $[C] = [\mu(A)] + q[\mu(B)]$ in $H_1(Y)$. When we glue $[(S^1 \times M_K) \setminus \nu T_m]$ to $S^1 \times Y$, the homology class $[\mu(B)]$ gets identified with $[\{\text{point}\} \times \lambda(K)] \in H_1((S^1 \times M_K) \setminus \nu T_m)$, which is trivial. Hence by Künneth's theorem, $[T_C] = [S^1 \times \mu(A)]$ in $H_2([S^1 \times Y] \cup [(S^1 \times M_K) \setminus \nu T_m])$. It follows that $[T_C] = [F]$ in $H_2(E(1)_K)$. \square

4. NON-ISOTOPY: SEIBERG-WITTEN INVARIANTS

Our strategy is to show that the isotopy types of the tori $\{T_C\}_{q \geq 1}$ can be distinguished by comparing the Seiberg-Witten invariants of the corresponding family of fiber sum 4-manifolds $\{E(1)_K \#_{T_C=F} E(n)\}_{q \geq 1, n \geq 1}$. Note that there is a canonical framing of a regular fiber F in $E(n)$, coming from the elliptic fibration $E(n) \rightarrow \mathbb{CP}^1$.

Lemma 4.1. *The fiber sum $E(1)_K \#_{T_C=F} E(n)$ is diffeomorphic to the link surgery manifold $L_q(\mathfrak{D}')$, where*

$$(4.1) \quad \mathfrak{D}' := (\{(\mu(A), \lambda(A)), (\lambda(B), -\mu(B)), (\lambda(C), -\mu(C))\},$$

$$\{(E(1), F = C_1 \times C_2), (S^1 \times M_K, T_m = S^1 \times \mu(K)), (E(n), F = C_1 \times C_2)\}).$$

Proof. We already observed in the proof of Lemma 2.4 that the fiber sum construction corresponds to this type of link surgery. (See also [EP2].) \square

Recall that the Seiberg-Witten invariant \overline{SW}_X of a 4-manifold X can be thought of as an element of the group ring of $H_2(X)$, i.e. $\overline{SW}_X \in \mathbb{Z}[H_2(X)]$. If we write $\overline{SW}_X = \sum_g a_g g$, then we say that $g \in H_2(X)$ is a Seiberg-Witten *basic class* of X if $a_g \neq 0$. Since the Seiberg-Witten invariant of a 4-manifold is a diffeomorphism invariant, so are the divisibilities of Seiberg-Witten basic classes. The Seiberg-Witten invariant of the link surgery manifold $L_q(\mathfrak{D}')$ is known to be related to the Alexander polynomial Δ_{L_q} of the link L_q .

Lemma 4.2. $\Delta_{L_q}(x, s, t) = 1 - x(st)^q$, where the variables x , s and t correspond to the components A , B and C respectively.

Proof. This follows readily from the formula in Theorem 1 of [Mo] which gives the multivariable Alexander polynomial of a braid closure and its axis in terms of the representation of the braid. We view A as the axis of the closure of a 2-strand braid, $B \cup C$. See [EP1] for details on a similar computation. \square

Theorem 4.3. *Let $\iota : [S^1 \times (S^3 \setminus \nu L_q)] \rightarrow L_q(\mathfrak{D}')$ be the inclusion map. Let $\xi := \iota_*[S^1 \times \mu(A)]$, $\tau := \iota_*[S^1 \times \mu(C)] \in H_2(L_q(\mathfrak{D}'))$. Then ξ and τ are both primitive and linearly independent. The Seiberg-Witten invariant of $L_q(\mathfrak{D}')$ is given by*

$$(4.2) \quad \overline{SW}_{L_q(\mathfrak{D}')} = (\xi^{-1} - \xi)^{n-1} \cdot \Delta_K^{\text{sym}}(\xi^2 \tau^{2q}),$$

where Δ_K^{sym} is the symmetrized Alexander polynomial of the knot K .

Proof. Let $N := (S^3 \setminus \nu L_q)$, and let $Z := [(S^1 \times M_K) \setminus \nu T_m]$. Recall from [Pa] that we have $\overline{SW}_{E(n) \setminus \nu F} = ([F]^{-1} - [F])^{n-1}$, and also

$$\overline{SW}_Z^\pm = \overline{SW}_{(S^1 \times M_K) \setminus \nu T_m}^\pm = \frac{\Delta_K^{\text{sym}}([T_m]^2)}{[T_m]^{-1} - [T_m]}.$$

From the gluing formulas in [Pa] and [Ta], we may conclude that

$$\overline{SW}_{L_q(\mathfrak{D}')} = \overline{SW}_{E(1) \setminus \nu F} \cdot \overline{SW}_{E(n) \setminus \nu F} \cdot \overline{SW}_{(S^1 \times M_K) \setminus \nu T_m}^\pm \cdot \Delta_{L_q}^{\text{sym}}(\xi^2, \sigma^2, \tau^2),$$

where $\sigma := \iota_*[S^1 \times \mu(B)]$. Note that $\sigma = 1 \in \mathbb{Z}[H_2(L_q(\mathfrak{D}'))]$, since we have $[\mu(B)] = [\lambda(K)] = 0$ in $H_1(Z)$. Also note that $\mu(K)$ and $\lambda(B)$ are identified by the gluing data \mathfrak{D}' , and $[\lambda(B)] = [\mu(A)] + q[\mu(C)] \in H_1(N)$. It follows that $[T_m] = \iota_*[S^1 \times \lambda(B)] = \xi\tau^q \in \mathbb{Z}[H_2(L_q(\mathfrak{D}'))]$. Thus we have

$$\Delta_{L_q}^{\text{sym}}(\xi^2, \sigma^2, \tau^2) = \xi^{-1}\tau^{-q} - \xi\tau^q = [T_m]^{-1} - [T_m].$$

Hence

$$(4.3) \quad \overline{SW}_{L_q(\mathfrak{D}')} = ([F]^{-1} - [F])^{n-1} \cdot \Delta_K^{\text{sym}}(\xi^2\tau^{2q}).$$

Note that the fiber F in $E(n)$ gets identified with $S^1 \times \lambda(C)$ by the gluing data \mathfrak{D}' , and we also have $[\lambda(C)] = [\mu(A)] + q[\mu(B)] = [\mu(A)]$ in $H_1([S^1 \times N] \cup Z)$. Therefore we can identify $[F] = \xi$ in (4.3), and we obtain Equation (4.2).

Next we show that ξ and τ are primitive and linearly independent elements of $H_2(L_q(\mathfrak{D}'))$. We can proceed in two different ways. A Mayer-Vietoris argument, combined with Freedman's classification theorem (cf. [FQ]), shows that $L_q(\mathfrak{D}')$ is homeomorphic to $E(n+1)$. It is not too hard to find two closed surfaces R and S in $L_q(\mathfrak{D}')$ satisfying

$$\xi \cdot [S] = \tau \cdot [R] = 1,$$

and

$$\xi \cdot [R] = \tau \cdot [S] = [R] \cdot [S] = 0.$$

For example, we can let S be the internal tubular sum of punctured sections from $[E(1) \setminus \nu F]$ and $[E(n) \setminus \nu F]$ summands, together with a suitable punctured surface from the Z summand. Let R be the internal tubular sum of the self-intersection (-1) disks bounding the circle C_2 in $[E(1) \setminus \nu F]$ and $[E(n) \setminus \nu F]$ summands, together with a suitable punctured surface from the Z summand.

In $L_q(\mathfrak{D}')$, S plays the role of a section in $E(n+1)$, while ξ plays the role of the homology class of the fiber. Note that we have $[\mu(C)] = [\lambda(A)] - [\mu(B)] = [\lambda(A)] \in H_1([S^1 \times N] \cup Z)$, and the gluing data \mathfrak{D}' identifies $\lambda(A)$ with a meridian circle $\mu(F)$ of the fiber F in $\partial[E(1) \setminus \nu F]$. Hence τ plays the role of the homology class of the rim torus $C_1 \times \mu(F)$ in $E(n+1)$. R plays the role of a self-intersection (-2) sphere transversally intersecting the above rim torus once. The pairs $(\xi, [S])$ and $(\tau, [R])$ form homology bases for two $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ summands in the intersection form of $L_q(\mathfrak{D}')$.

Alternatively, we can argue more algebraically as follows. Consider the composition of homomorphisms

$$(4.4) \quad H_1(N) \longrightarrow H_2(S^1 \times N) \xrightarrow{\iota_*} H_2(L_q(\mathfrak{D}')),$$

where the first map is a part of the Künneth isomorphism

$$(4.5) \quad H_1(N) \oplus H_2(N) \xrightarrow{\cong} H_2(S^1 \times N).$$

Note that $H_2(N) \cong \mathbb{Z} \oplus \mathbb{Z}$, as is easily seen from the long exact sequence of the pair $(N, \partial N)$ as follows.

$$\begin{array}{ccccccccc} H^0(N) & \longrightarrow & H^0(\partial N) & \longrightarrow & H^1(N, \partial N) & \xrightarrow{0} & H^1(N) & \longrightarrow & H^1(\partial N) \\ \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & H_2(N) & \xrightarrow{0} & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^6 \end{array}$$

Note that the first map sends the generator $1 \in \mathbb{Z}$ to the diagonal element $(1, 1, 1) \in \mathbb{Z}^3$, while the last map is injective. We have also used the Lefschetz duality theorem (for manifolds with boundary) to identify $H_2(N) \cong H^1(N, \partial N)$.

Next consider the long exact sequence of the pair $(L_q(\mathfrak{D}'), S^1 \times N)$:

$$0 = H_3(L_q(\mathfrak{D}')) \longrightarrow H_3(L_q(\mathfrak{D}'), S^1 \times N) \longrightarrow H_2(S^1 \times N) \xrightarrow{\iota_*} H_2(L_q(\mathfrak{D}'))$$

The kernel of the last map ι_* is isomorphic to $H_3(L_q(\mathfrak{D}'), S^1 \times N)$. By Lefschetz duality theorem (for relative manifolds), $H_3(L_q(\mathfrak{D}'), S^1 \times N)$ is in turn isomorphic to

$$H^1(L_q(\mathfrak{D}') \setminus (S^1 \times N)) \cong H^1(E(1) \setminus \nu F) \oplus H^1(E(n) \setminus \nu F) \oplus H^1(Z).$$

Since we have $H^1(E(1) \setminus \nu F) = H^1(E(n) \setminus \nu F) = 0$ and

$$(4.6) \quad H^1(Z) = H^1(S^1 \times (M_K \setminus \nu m)) \cong \mathbb{Z} \oplus \mathbb{Z},$$

the kernel of ι_* is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Finally we observe that only one \mathbb{Z} summand of (4.6) lies in the kernel of the composition (4.4). The other \mathbb{Z} summand belongs to the kernel of

$$H_2(N) \longrightarrow H_2(S^1 \times N) \xrightarrow{\iota_*} H_2(L_q(\mathfrak{D}')),$$

where the first map is the second part of the Künneth isomorphism (4.5). We have thus shown that the kernel of the composition (4.4) is of rank one. It follows immediately that ξ and τ are linearly independent, since we already have shown that σ is trivial. A more detailed analysis shows that $\{\xi, \tau\}$ can be extended to a basis of $H_2(L_q(\mathfrak{D}'))$, which we shall omit. (Also see the proof of Proposition 3.2 in [MT] for a similar argument.) \square

Corollary 4.4. *If K is a nontrivial fibred knot, then the tori $\{T_C\}_{q \geq 1}$ are pairwise non-isotopic inside $E(1)_K$. In fact, there is no self-diffeomorphism of $E(1)_K$ that maps one element of this family to another.*

Proof. Let's choose n to be $2g + 1$, where g is the genus of K . Remember that the degree of the symmetrized Alexander polynomial of a fibred knot is the same as its genus (see e.g. Proposition 8.16 in [BZ]). Since we assume that K is nontrivial, i.e. not the unknot, $g > 0$. A Seiberg-Witten basic class of $L_q(\mathfrak{D}')$ with the highest divisibility is divisible by $2gq$. This could be seen by observing that the highest power of τ in (4.2) of Theorem 4.3 is $2gq$ (hence there cannot be a basic class with divisibility higher than $2gq$) and our choice of $n = 2g + 1$ ensures that there is a basic class (namely τ^{2gq}) with this highest possible divisibility. On the other hand, since the Seiberg-Witten invariant is a diffeomorphism invariant, so are the divisibilities of basic classes. Therefore, $L_q(\mathfrak{D}')$ is diffeomorphic to $L_{q'}(\mathfrak{D}')$ if and only if $q = q'$ proving that the tori in $\{T_C\}_{q \geq 1}$ are different up to isotopy and in fact even up to self-diffeomorphisms of $E(1)_K$. \square

This concludes the proof of Theorem 1.1.

5. GENERALIZATION TO OTHER SYMPLECTIC 4-MANIFOLDS

When K is the unknot, $E(1)_K$ is diffeomorphic to $E(1)$. In this unknot case, our family of tori $\{T_C\}_{q \geq 1}$ are easily seen to be all isotopic to one another. The isotopy can actually be visualized by erasing the B component in Figure 2 (This corresponds to filling in νB with $(M_K \setminus \nu m)$, which, in the unknot case, is diffeomorphic to a solid torus $S^1 \times D^2$), and straightening out the C component through the tubular neighbourhood of B , which has now been filled in. Note that the normal disks of B are the Seifert surfaces of the unknot.

Suppose that K is not fibred. Then, unlike the fibred case where the degree of the Alexander polynomial is (the same as the genus hence) strictly greater than 0 unless the knot is the unknot, the Alexander polynomial of K might be constant and the Seiberg-Witten invariant doesn't seem to be delicate enough to distinguish the tori we constructed. On the other hand, when K is not fibred and has a non-constant Alexander polynomial, the tori in our family $\{T_C\}_{q \geq 1}$ are still pairwise non-isotopic in $E(1)_K$, but there is no natural symplectic structure on $E(1)_K$ and we don't know whether T_C is symplectic with respect to a symplectic structure on $E(1)_K$. In fact, it is known that $E(1)_K$ doesn't admit any symplectic structure if the Alexander polynomial of K is not monic [FS1].

On a more positive note, we can easily extend Theorem 1.1 to $E(n)_K$ ($n \geq 2$) and more generally to X_K , where X is a symplectic 4-manifold satisfying certain topological conditions as in [EP1].

Proposition 5.1. *Assume that F is a symplectic 2-torus in a symplectic 4-manifold X . Suppose that $[F] \in H_2(X)$ is primitive, $[F] \cdot [F] = 0$, and $H^1(X \setminus \nu F) = 0$. If $b_2^+(X) = 1$, then we also assume that $\overline{SW}_{X \setminus \nu F} \neq 0$ and is a finite sum. Then there exists an infinite family of pairwise non-isotopic symplectic tori in X_K representing the homology class $[F] \in H_2(X_K)$ for any nontrivial fibred knot $K \subset S^3$.*

The divisibility argument in the proof of Corollary 4.4 may not work in this general setting, but after observing that an isotopy between these tori should preserve ξ and τ , one can resort to a homology basis argument due to Fintushel and Stern which was announced in [FS4].

It may be possible, as in the rational elliptic surface case, to show that these non-isotopic tori are inequivalent under self-diffeomorphisms of X_K once we know the Seiberg-Witten invariant of $[X \setminus \nu F]$ explicitly, but a general argument doesn't seem to exist at this moment.

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